ISRAEL JOURNAL OF MATHEMATICS 102 (1997), 297–315

NEIGHBORLY CUBICAL SPHERES AND A CUBICAL LOWER BOUND CONJECTURE*

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ABSTRACT

Using mirrors and cyclic polytopes, we construct cubical d-spheres which are the analogs of cyclic polytopes in the sense that they have the $\lfloor \frac{d-1}{2} \rfloor$-skeleta of cubes. The existence of these neighborly cubical spheres leads to a special case of an upper bound conjecture for cubical spheres, suggested by Kalai. We extend the same construction to show that the closed convex hull of f-vectors of cubical spheres contains a cone described by Adin, as an analog to the generalized lower bound theorem for simplicial polytopes.

* Supported in part, respectively, by an NSF Postdoctoral Fellowship; NSF Grants DMS 9207700 and 9500581; and an NSF Planning Grant.
Received September 13, 1995 and in revised form May 6, 1996

1. Introduction

In the past few years, there has been much activity regarding the enumeration of faces of cubical polytopes. In this paper we continue efforts to put this subject on more of a parallel track with that of simplicial polytopes.

In §2, we discuss some basic constructions of cubical complexes—mirrors, fissures and barycentric covers. In §3 we use the mirror and fissure operations to produce what we call "neighborly" cubical spheres. In §4, we define Adin's "cubical h-vector", an enumerative invariant for cubical complexes [1], and use it to prove a special case of an upper bound conjecture due to Kalai.

In §5 we consider a cubical analog of the generalized lower bound theorem for simplicial polytopes, formulated in terms of Adin's cubical h-vector. We show that if this conjecture holds for all cubical spheres, it gives the tightest set of linear inequalities possible for their face numbers. We conclude with questions in §6.

We would like to thank Rachel Hastings, Gabor Hetyei and an anonymous referee for reading and making useful comments on earlier versions of this paper.

Some preliminaries on posets are in order here. For a poset P, we denote by $[x,y]$ or $[x,y]_P$ the interval $\{z \in P\colon x \le z \le y\}$, by $\bigwedge(x)$ or $\bigwedge_P(x)$ the principal (lower) **order ideal** $\{z \in P\colon z \le x\}$, and by $\bigvee(x)$ or $\bigvee_P(x)$ the principal **filter** $\{z \in P\colon z \ge x\}$. We will also refer to $\bigvee_P(x)$ as the link in P of x or $\mathrm{lk}_P\ x$. By $\widehat{P}$ we mean P with a 0 and 1 adjoined, and by P^{op} we mean the underlying set of P with the order reversed. We denote by $|P|$ the (simplicial) complex of chains in P.

By a **simplicial poset** we mean one in which every order ideal $\bigwedge(x)$ is a Boolean algebra (i.e., a product of copies of B_1, the Boolean algebra on one element). By a **cubical poset** we mean one in which each order ideal $\bigwedge(x)$ is a product of copies of I, the face poset of an interval, excluding the empty set. We consider I to be the poset $\{0,1,-1\}$ with ordering $1 < 0$ and $-1 < 0$ (and so the Hasse diagram of I is $\bigwedge$). Thus the face poset of any simplicial complex (including the empty set) is simplicial, while the face poset of any cubical complex (excluding the empty set) is cubical. For this reason, throughout this paper when we consider the face poset of any simplicial complex, it will always **include** the empty set, while that of a cubical complex will always **exclude** the empty set. Two other concepts we will use are the boundary and interior of a poset. If P is a finite poset, denote by $\partial P := \{y\colon y \le x, |\bigvee_P(x)| = 2\}$ the boundary of P, and call $P^\circ := P\backslash\partial P$ the interior of P. To keep notation to a minimum, we will usually denote a complex and its face poset by the same symbol.

Both simplicial and cubical posets are ranked, the rank of an element being one less than the cardinality of a maximal chain ending at this element. Thus in a simplicial complex, the rank of a face is one more than its dimension, while in a cubical complex, rank is the same as dimension. We will restrict our attention here to simplicial posets with unique minimal element that are meet-semilattices (i.e., simplicial complexes) and to cubical posets P such that $\widehat{P}$ is a lattice (called cubical complexes). A poset map will be called a complex map if it preserves rank. Thus being a subcomplex of a cubical complex is a stronger property than merely being a subposet. We note that poset product corresponds to complex join in the simplicial case and product in the cubical case, and thus order ideals in cubical posets are posets of cubes. Finally, for a complex X, we denote by $(X)_k$ its k-skeleton (the set of all r-faces of X, $r \leq k$).

2. Mirrors, fissures and barycentric covers

We define and study three constructions leading to cubical complexes. Two of these, mirroring and fissuring, are used in the constructions in later sections.

2.1 MIRRORING. We begin with an operation which converts a simplicial complex to a cubical complex. Let T be a subcomplex of the $(n-1)$-simplex σ^{n-1}. We can think of T (including the empty set) as a partially ordered set (the corresponding simplicial poset) where each face is a 0-1 vector with a 0 in the i^{th} place if and only if the vertex i belongs to the face. Thus T is partially ordered by $0 > 1$ extended componentwise. Then we construct a partially ordered set MT as follows:

$$MT := \{(a_1, a_2, \ldots, a_n) \colon (|a_1|, |a_2|, \ldots, |a_n|) \in T\} \subseteq I^n,$$

partially ordered by $0 > 1$ and $0 > -1$ extended componentwise. Note that MT depends on the ambient simplex σ^{n-1} as well as the complex T.

This operation has a long history. For the case of the m-gon, it was used by Coxeter [6] to produce regular maps $\{4, m|4^{[\frac{1}{2}m]-1}\}$ on surfaces. It has been used by Davis in the study of reflection groups and toric varieties (see [7, p. 108], for the Coxeter system $(\mathbf{Z}_2^{[n]}, [n])$, and [8]). It has also been studied by Schulte [15, §5], where it is denoted 2^T, and in [12, §3C]. A dual version, as illustrated in Figure 2, can be found in [4, §3.2]; indeed, using the notation there, $MT = (\mathcal{B}_{T^{op}})^{op}$, where the subspace arrangement is a poset with ordering by inclusion.

A simple example is given in Figure 1. Here T is the simplicial complex consisting of two adjacent edges, and MT is the boundary of the 3-cube minus two opposite (open) facets.

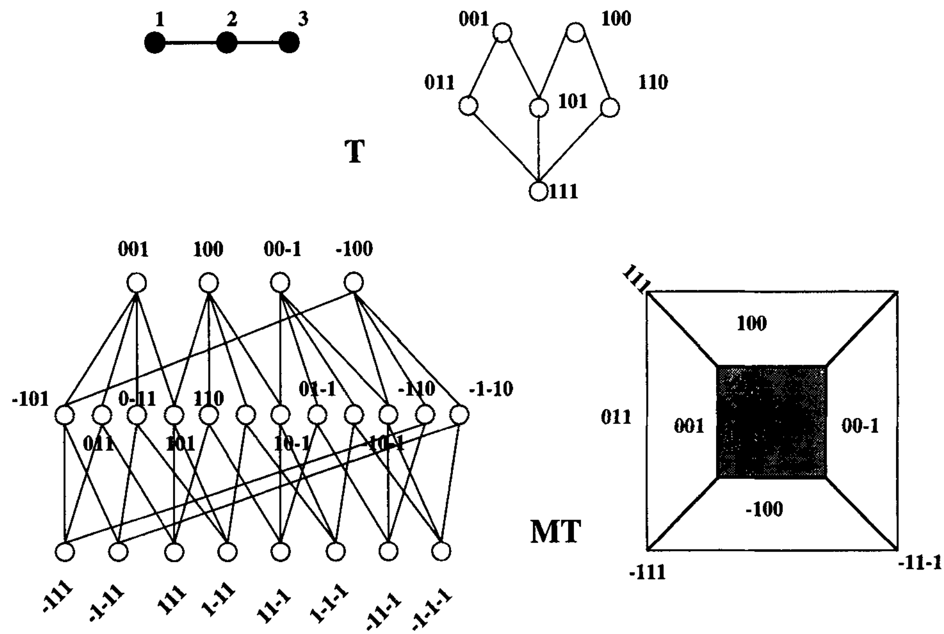


Figure 1. The mirror complex of two edges.

PROPOSITION 2.1: *MT is the face lattice of a cubical complex in which the link of any vertex is isomorphic to the original simplicial complex T. If T has the k-skeleton of the $(n-1)$-simplex, then MT has the $(k+1)$-skeleton of the n-cube.*

Proof: To see that MT is a cubical complex, note that it can be realized as a subcomplex of the n-cube $[-1,1]^n$ by associating the point $a = (a_1, a_2, \ldots, a_n)$ with the face of the cube having a as its centroid. The interval lying above any minimal element of MT is, up to signs, the poset T. The statement about skeletons follows directly. ∎

It follows immediately that if T is a d-simplex, then MT is a $(d+1)$-cube. Further, $\partial(MT) = M(\partial T)$.

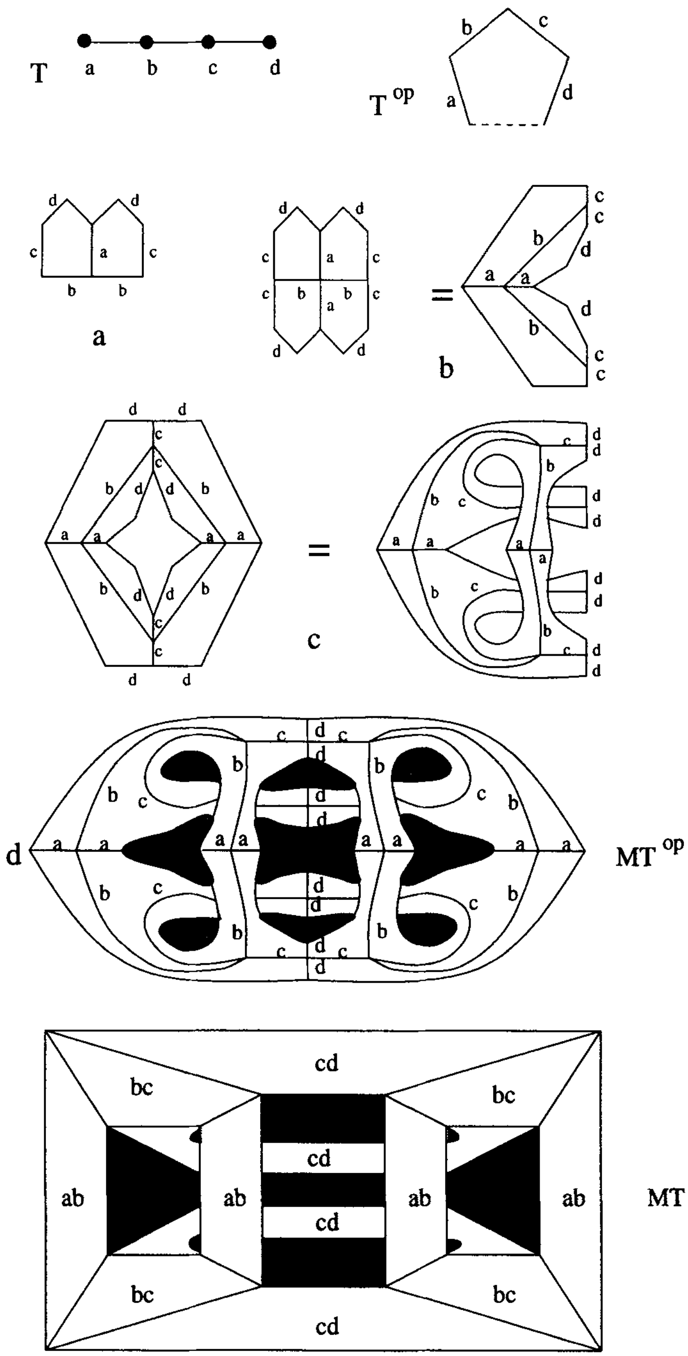


Figure 2. The mirror complex of three edges.

We say MT results from a mirroring of T, since if T is the boundary complex of a simplicial polytope P then by taking the dual P^{op} of P, mirroring P^{op} across all its facets, and then taking the dual of the resulting cell complex, we get back MT. This also works for more general T, as is illustrated in Figure 2, where T is a line segment divided into three edges.

We note here that since the operation M commutes with poset product, we get $M(T_1 * T_2) = M(T_1) \times M(T_2)$, where $T_1 * T_2$ is the join of complexes T_1 and T_2. Further note that if T is a subcomplex of σ^{n-1}, the f-polynomials (see (6)) of T and MT are related by

$$f(MT,t) = 2^n f(T,t/2). \tag{1}$$

2.2 Cubical fissures. We define next an operation on a cubical poset C that depends on a pair of order ideals C_1 and C_2 in C. Let $C(C_1, C_2) \subset C \times I$ be the poset defined by

$$C(C_1,C_2) := (C_1 \times \{1\}) \cup (C_1 \cap C_2 \times \{0\}) \cup (C_2 \times \{-1\}). \tag{2}$$

We call this the **fissure** of C **between** C_1 and C_2 (or **along** $C_1 \cap C_2$).

That $C(C_1, C_2)$ is cubical follows from the fact that it is an order ideal in the cubical poset $C \times I$. Topologically, we have the relation

$$\begin{aligned} |C(C_1,C_2)| = |C_1| \cup_{C_1\cap C_2\times\{1\}} \Big(|C_1 \cap C_2| \times [-1,1]\Big) \\ \cup_{C_1\cap C_2\times\{-1\}} \; |C_2|. \end{aligned} \tag{3}$$

When C is the poset of a cubical complex (also denoted by C), $C(C_1, C_2)$ is the poset of the complex obtained by lifting C_1 by height one, dropping C_2 by one, and filling in the resulting fissure by $(C_1 \cap C_2) \times [-1, 1]$. See Figure 3.

We can iterate the fissuring of C between a pair of complexes as follows. Let $C(C_1,C_2)^0 := C_1 \cup C_2$, $C(C_1,C_2)^1 := C(C_1,C_2)$ and, for $r \geq 1$,

$$C(C_1,C_2)^r := \Big(C(C_1,C_2)^{r-1}\Big)\Big(C_1(C_1, C_1 \cap C_2)^{r-1}, C_2\Big). \tag{4}$$

Note that successive fissurings separate C_1 and C_2 by more and more copies of $(C_1 \cap C_2) \times [-1, 1]$. It follows directly from (2) and (4) that

$$f(C(C_1,C_2)^r,t) = f(C_1 \cup C_2,t) + r(1+t)f(C_1 \cap C_2,t). \tag{5}$$

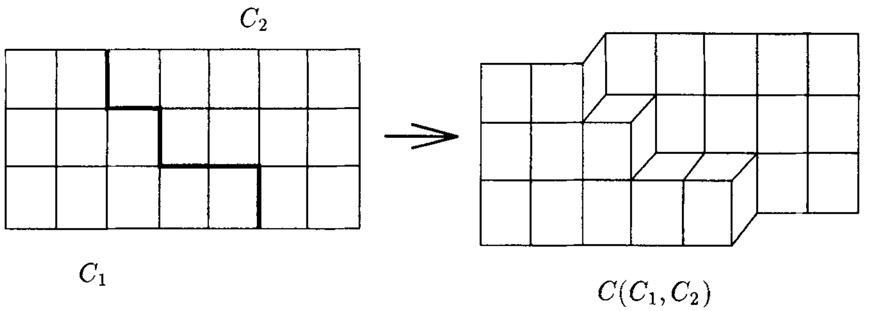


Figure 3. Fissuring along $C_1 \cap C_2$.

2.3 BARYCENTRIC COVERS. Finally, we define a cubical complex midway between a complex of **simple** polytopes and its barycentric subdivision. We call it the **barycentric cover**. In the case of a simplicial complex, this is the same as the **cubical barycentric subdivision** used by Hetyei [10]. The remainder of this section is not used in what follows.

Let KP be the poset with elements the order relations of an arbitrary poset P, partially ordered by inclusion (i.e., $(u \leq v) \leq (x \leq y)$ if and only if $x \leq u \leq v \leq y$). As shown below, KP is cubical whenever P is a poset having all intervals Boolean algebras. This includes simplicial and cubical posets (more generally, face posets of polyhedral complexes with simple, nonempty cells) as well as their duals. In fact $KP = K(P^{op})$. It is straightforward to check that K distributes over product, i.e., $K(P \times Q) = K(P) \times K(Q)$.

PROPOSITION 2.2: *If P has Boolean intervals then KP is a cubical poset. Further, if $\widehat{P}$ is a lattice, then so is $\widehat{KP}$.*

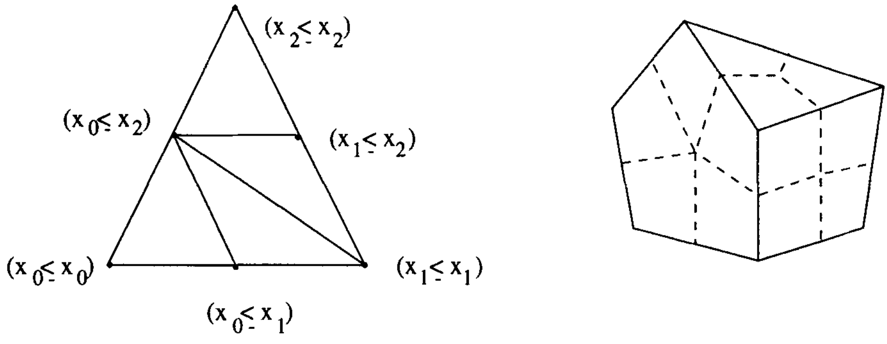
THEOREM 2.3: *$|KP|$ gives a polyhedral subdivision of $|P|$ by the map taking the point $(x \leq x)$ to itself and the point $(x < y)$ in $|KP|$ to the midpoint of the edge $x < y$ in $|P|$, and extending linearly over every closed simplex in $|KP|$.*

In the case in which P is itself the face poset of a polyhedral complex, we obtain the following corollary, which justifies calling KP the **barycentric cover** of P.

COROLLARY 2.4: *If P is the face poset of a polyhedral complex, then KP is the face poset of a polyhedral subdivision of P, lying between P and $|P|$ in the refinement order.*

When the underlying complex of P has simple facets and P does not include the empty set (so P has Boolean intervals), KP will be a cubical complex by

Proposition 2.2. See Figure 4(b). Further, the triangulation of each d-cube in KP induced by $|P|$ is the standard triangulation into $d!$ simplices given by all the coordinate permutations.



(a) (b)

Figure 4. (a) The subdivision of the simplex $x_0 \leq x_1 \leq x_2$.
(b) The barycentric cover of a complex with three maximal cells.

3. Neighborly cubical spheres

Now we use the mirroring and fissuring operations to produce a "neighborly" cubical sphere for each $n \geq d+1$, i.e., a cubical d-sphere with the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of the n-cube. The existence of such spheres was suggested by Kalai (personal communication). We begin by constructing, for a given simplicial d-polytope P, a cubical $(d+1)$-sphere having the mirror complex of ∂P as a subcomplex.

THEOREM 3.1: *If P is a simplicial d-polytope, then there is a triangulation K of P and a PL-cubical $(d+1)$-sphere C such that the mirror complex MK is a subcomplex of C having $M\partial P$ as its boundary.*

Proof: Given the polytope P, along with an arbitrary ordering on its vertices $v_0, v_1, \ldots, v_{n+d}$ (which we assume are in general position), we form a sequence of simplicial d-balls $K_0, \ldots, K_n$ such that $K_i \subset \partial K_{i-1} * \{v_{d+i}\}$, where K_0 is the d-simplex spanned by $v_0, \ldots, v_d$ and K_i is the join of v_{d+i} with that part of the boundary of K_{i-1} that it does not see. By construction, the K_i are all PL d-balls and $\partial K_n = \partial P$, so $\partial MK_n = M\partial P$.

Let C_0 be the cubical poset made up of two $(d+1)$-cubes sharing a common boundary. We consider the $(d+1)$-cube MK_0 to be one of these cubes, and hence $C_0 \setminus (MK_0)^\circ$ is the other. Finally, $|C_0|$ is a PL $(d+1)$-sphere.

Define

$$C_{i+1} := C_i(MK_i, C_i \setminus (MK_i)^\circ).$$

We assert that, for $1 \leq i < n$, C_{i+1} is well-defined and is a cubical PL $(d+1)$-sphere. To verify that C_{i+1} is well-defined, we must check that both MK_i and $C_i \setminus (MK_i)^\circ$ are subcomplexes (i.e., order ideals) in C_i. Assuming this to be true for C_{i-1} and MK_{i-1}, we observe that for $i \geq 1$

$$\begin{aligned} MK_i \subset & M(\partial K_{i-1} * \{v_{d+i}\}) \\ =& M\partial K_{i-1} \times I \\ =& \partial MK_{i-1} \times I \\ \subset & C_{i-1}(MK_{i-1}, C_{i-1} \setminus (MK_{i-1})^\circ) \\ =& C_i. \end{aligned}$$

Note that beginning with C_1, all the C_i are cubical **complexes** and that both the inclusions above are inclusions of complexes (preserve rank). This shows MK_i to be a subcomplex of C_i.

To verify that $C_i \setminus (MK_i)^\circ$ is an order ideal in C_i, we note, by induction, that MK_i is a full-dimensional (pure) subcomplex in the the sphere C_i, and so the complement of its interior is a complex. Finally, to check that C_{i+1} is a PL $(d+1)$-sphere, note that MK_i is a PL $(d+1)$-manifold with boundary, since its links are links in K_i. Thus ∂MK_i is collared in MK_i [14, Corollary 2.26], and hence by (3), it follows that $C_{i+1} \cong_{\mathrm{PL}} C_i$.

Now take $C := C_n$ and $K := K_n$. ∎

Figure 5 illustrates the construction of the 3-sphere C corresponding to a pentagon. C_0 is the 3-sphere consisting of two 3-cubes joined along their boundaries, and MK_0 is the "inside" cube. C_1 is then the boundary of the 4-cube. Since $K_1 = T * v_3$, where T is the complex of two edges, $MK_1 = MT \times I$, where MT is as given in Figure 1. The final complex MK (not pictured) is the product of I with the cubical complex shown in Figure 2.

A first consequence of Theorem 3.1 and its proof is the existence of "neighborly" cubical spheres.

Definition 3.2: A cubical complex is said to be n-**neighborly** if its n-skeleton is that of a cube.

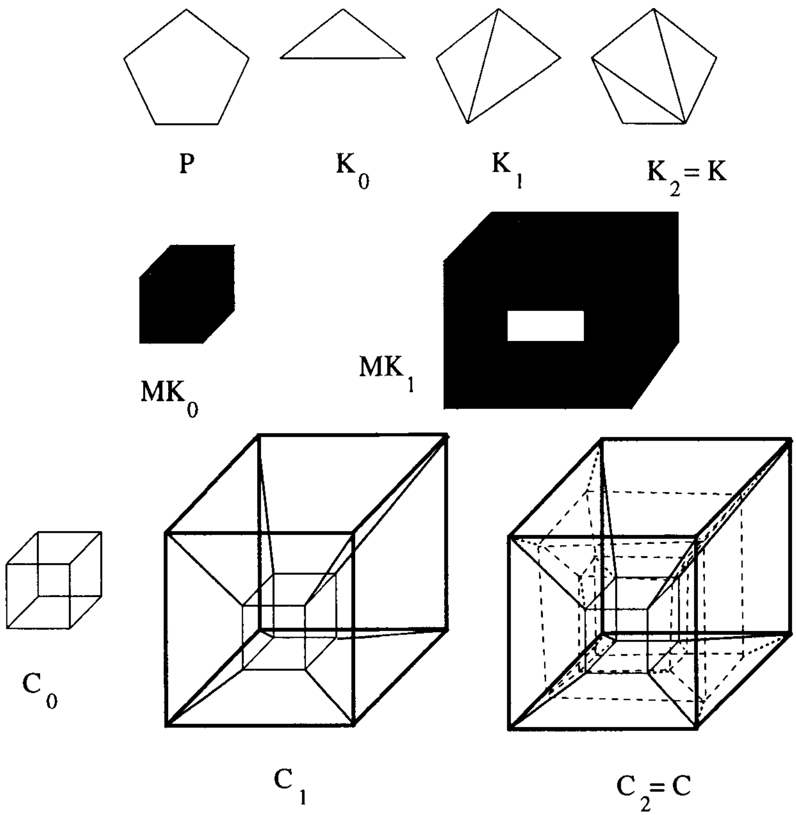


Figure 5. A cubical 3-sphere C from a pentagon P.

COROLLARY 3.3: *There exist $\lfloor \frac{d-1}{2} \rfloor$-neighborly cubical d-spheres with 2^k vertices for every $k > d$.*

Proof: Choose $P := C(k, d-1)$, the cyclic $(d-1)$-polytope with k vertices, ordered arbitrarily. Note that each ball K_i in the proof of Theorem 3.1 has as boundary the cyclic polytope $C(d+i, d-1)$. Hence, each K_i has as its $\lfloor \frac{d-3}{2} \rfloor$-skeleton the $\lfloor \frac{d-3}{2} \rfloor$-skeleton of a $(d+i-1)$-simplex. Thus the $\lfloor \frac{d-1}{2} \rfloor$-skeleton of $M\partial K_i$ is that of a $(d+i)$-cube.

We will show, by induction, that $(C_i)_{\lfloor \frac{d-1}{2} \rfloor} = (M\partial K_i)_{\lfloor \frac{d-1}{2} \rfloor}$. First, recall that C_0 has the $(d-1)$-skeleton of a d-cube, and $(C_0)_{d-1} = (M\partial K_0)_{d-1}$. Assuming the assertion for i, it follows from the fissure construction that

$$\begin{aligned}(C_{i+1})_{\lfloor\frac{d-1}{2}\rfloor} &= \Big(C_i(MK_i, C_i\setminus(MK_i)^\circ)\Big)_{\lfloor\frac{d-1}{2}\rfloor}\\ &= \Big(M\partial K_i \times I\Big)_{\lfloor\frac{d-1}{2}\rfloor}\\ &= (I^{d+i+1})_{\lfloor\frac{d-1}{2}\rfloor}\\ &= \Big(M\partial K_{i+1}\Big)_{\lfloor\frac{d-1}{2}\rfloor}.\end{aligned}$$

The second equality follows from the fact that the fissuring is taking place along $M\partial K_i$, which has the same $\lfloor\frac{d-1}{2}\rfloor$-skeleton as C_i by the induction hypothesis.

Thus, C_{k-d-1} is the desired $\lfloor\frac{d-1}{2}\rfloor$-neighborly cubical d-sphere with 2^k vertices. ∎

We remark that the spheres constructed in Corollary 3.3 are always PL. It is an open question whether there exist neighborly **polytopal** spheres.

4. Adin's h-vector and Kalai's upper bound conjecture

Recently, Adin defined a "cubical h-vector" for studying the face numbers of cubical complexes [1]. This invariant appears to be a good analog of the usual h-vector for simplicial complexes.

For a ranked poset P, with $\text{rank}(P) = d$, we denote by $f_i := f_i(P)$ the number of elements of rank i and define the polynomial

$$f(P,t) := \sum_{i=0}^{d} f_i t^i. \tag{6}$$

From this, we define polynomials $h^s(P,t)$, $h^{sc}(P,t)$ and $h^c(P,t)$ by

$$h^s(P,t) = (1-t)^d f\left(P, \frac{t}{1-t}\right), \tag{7}$$

$$h^{sc}(P,t) = (1-t)^d f\left(P, \frac{2t}{1-t}\right) \tag{8}$$

and

$$h^c(P,t) = \frac{t(1-t)^d}{1+t} f\left(P, \frac{2t}{1-t}\right) + 2^d \frac{1+\tilde{\chi}_P(-t)^{d+2}}{1+t} \tag{9}$$

$$= \frac{1}{1+t}\left(t\, h^{sc}(P,t) + 2^d(1+\tilde{\chi}_P(-t)^{d+2})\right), \tag{10}$$

where $\tilde{\chi}_P := f(P,-1) - 1$ is the reduced Euler characteristic of P (cf. [1, (1-3)]).

In general, the coefficient of t^i in a polynomial $q(t)$ will be denoted by q_i. The coefficients of h^s, h^{sc} and h^c will be referred to, respectively, as the **simplicial, short cubical** and **cubical h-vectors** of the ranked poset P. (Note that in h^s, one uses the simplicial rank in the f-polynomial, whereas in h^{sc} and h^c one uses the cubical rank.) As observed by Hetyei,

$$h^{sc}(K,t) = \sum_{v \in V} h^s(lk_K\, v, t).$$

It is also useful to observe that $h_i^{sc} = h_i^c + h_{i+1}^c$ for all $0 \le i \le d$ [1].

Each h-polynomial has an associated g-polynomial, defined by multiplying it by $1-t$. The associated "g-vector" is usually taken to be the first half of the coefficients of the g-polynomial (excluding the middle term, when the degree is even, as well as the constant term). Thus, if P is the boundary complex of a cubical d-polytope, the degrees of h^{sc}, h^c and g^c are $d-1$, d and $d+1$, respectively, so the relevant coefficients of g^c are $g_1^c, \ldots, g_{\lfloor \frac{d}{2} \rfloor}^c$.

Adin has shown that the cubical h-vector has many properties analogous to those of the simplicial h-vector. For example, if K is Eulerian then $h^c(K)$ is symmetric, and for any K, $h^c(K)$ is a lower-triangular linear transformation of $f(K)$. In particular, for any cubical $(d-1)$-complex K, and for all $i \le d$, we have

$$h_i^c(K) = (-1)^i 2^{d-1} f_{-1}(K) + \sum_{j=1}^{i} (-1)^{i-j} 2^{j-1} f_{j-1}(K) \sum_{k=0}^{i-j} \binom{d-j}{k}, \tag{11}$$

where $f_{-1}(K) := 1$ [1, Lemma 1]. The relation (11) can be inverted to give the f_j as nonnegative linear combinations of the h_i^c.

We state without proof a few more such properties of h^c. In what follows, we assume that K is a cubical complex homeomorphic to a $(d-1)$-ball. For $K^\circ = K \setminus \partial K$, we have

$$f(K^\circ, t) = f(K,t) - f(\partial K, t),$$

and we define h^{sc} for K° as in (8), with the same rank as K, namely, $d-1$, and h^c as in (11), but with $f_{-1}(K^\circ) := 0$.

Proposition 4.1: *If K is a cubical $(d-1)$-ball, then*

1. $h_i^c(K) = h_{d-i}^c(K^\circ)$ *for all i, and*
2. $g_i^c(\partial K) = h_i^c(K) - h_{d-i}^c(K)$ *for all $i \ge 1$.*

We now consider Kalai's upper bound conjecture for cubical spheres. Let C_r be a $\lfloor\frac{d-1}{2}\rfloor$-neighborly cubical d-sphere with 2^r vertices. By definition, $f_i(C_r)$ is the number of i-faces in an r-cube, for $i \leq \lfloor\frac{d-1}{2}\rfloor$. By the cubical Dehn–Sommerville equations, the remaining $f_i(C_r)$ are determined and thus are independent of the particular C_r chosen.

CONJECTURE 4.2 (Kalai): *If C is a cubical d-sphere with 2^r vertices, then $f_i(C) \leq f_i(C_r)$, for all i.*

Using the Adin h-vector it is easy to prove the conjecture in the case of odd d for any cubical d-sphere whose 1-skeleton lies in the r-cube.

THEOREM 4.3: *If d is odd and C is any cubical d-sphere for which every vertex has degree at most r, then $f_i(C) \leq f_i(C_r)$ for all i.*

Proof: If C is any such cubical d-sphere C , then $\mathrm{lk}_C\, v$ is a simplicial $(d-1)$-sphere with at most r vertices, for each v. Thus by the upper bound theorem for simplicial spheres, $h^s(\mathrm{lk}_C\, v) \leq h^s(S)$, where S is any $\lfloor\frac{d}{2}\rfloor$-neighborly simplicial $(d-1)$-sphere with r vertices. Since d is odd, $\lfloor\frac{d}{2}\rfloor = \lfloor\frac{d-1}{2}\rfloor$. Thus

$$h^{sc}(C) = \sum_v h^s(lk_C v) \leq \sum_v h^s(S) = h^{sc}(C_r),$$

since C_r has the $\lfloor\frac{d-1}{2}\rfloor$-skeleton of the r-cube. Thus also $f(C) \leq f(C_r)$, since the coefficients f_j are nonnegative linear combinations of the coefficients of h^{sc}. ∎

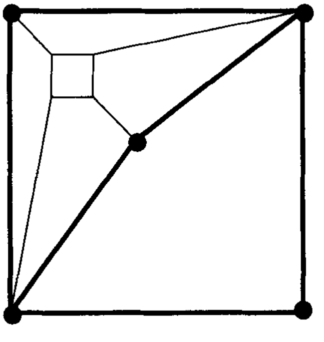


Figure 6. A cubical 2-ball containing $K_{2,3}$.

For each $d \geq 1$, it is easy to find an example of a cubical d-sphere whose 1-skeleton does not lie in the 1-skeleton of any cube. Note first that the complete bipartite graph $K_{2,3}$ is not a subcomplex of any cube. This is clear because any two vertices in a cube that are joined by a path of length 2 must differ in exactly two coordinates, and so there must be precisely two such paths. Next consider the cubical 2-ball C in Figure 6. Note that $K_{2,3}$ is a subgraph of $(C)_1$, as indicated. For $d \geq 2$, take $S^d := \partial(C \times I^{d-1}) \supset C$. For $d = 1$ simply choose the boundary of a triangle.

5. A cubical lower bound conjecture

Adin raised the following "generalized lower bound conjecture" as a question [1, Question 2].

CONJECTURE 5.1: *If K is a cubical d-polytope, then $g_i^c(\partial K) \geq 0$, for all $i \leq \lfloor \frac{d}{2} \rfloor$.*

Here we conjecture that these are the best possible linear inequalities.

CONJECTURE 5.2: *The closed convex hull of the f-vectors of all cubical d-polytopes is the translated cone given by the inequalities $g_i^c \geq 0$, for all $i \leq \lfloor \frac{d}{2} \rfloor$, and $g_0^c = 2^{d-1}$.*

Let e_i denote the positive ray in the i^{th} coordinate direction. One approach to proving Conjecture 5.2 is to show that, for any $1 \leq i \leq d/2$, there exist cubical d-polytopes with g^c arbitrarily close in direction to the ray e_i. If both conjectures are correct, then the "Adin g-cone" so defined (i.e., the set of f-vectors whose corresponding g^c-vectors are nonnegative) is exactly the closure of the convex hull of all f-vectors of cubical polytopes. The first conjecture is only known to be true in the case $i = 1$ [1, 5]. We address the second conjecture here.

5.1 STACKED CUBICAL POLYTOPES.

As a first attempt, we consider **stacked cubical polytopes**, analogous to the stacked simplicial polytopes considered by McMullen and Walkup in their paper introducing the generalized lower bound conjecture for simplicial polytopes [13].

Definition 5.3: A cubical d-polytope is **k-stacked** if its boundary is the boundary of a cubical ball with no interior $(d-1-k)$-faces. Similarly, a cubical d-ball is **k-stacked** if it has no interior $(d-1-k)$-faces.

Definition 5.4: A simplicial complex is called **k-neighborly** if every set of vertices of cardinality k is a face.

Neighborly stacked polytopes were used in [13, Theorem 3] to produce examples of simplicial polytopes having a simplicial g-vector with a dominant coordinate. The following shows that the cubical g-vector behaves analogously.

PROPOSITION 5.5: *If $k \leq d/2$ and $\{P_n\}$ is a sequence of k-stacked cubical d-polytopes such that $f_{k-1}(\partial P_n)$ dominates $f_i(\partial P_n)$ for all $i < k-1$, that is, for each such i,*

$$\frac{f_i(\partial P_n)}{f_{k-1}(\partial P_n)} \to 0 \quad \text{as } n \to \infty,$$

then $g^c_k(\partial P_n)$ dominates $g^c_i(\partial P_n)$ for all $i \neq k$.

Proof: For each n, if P_n is k-stacked, then $\partial P_n = \partial K_n$, where K_n is a cubical ball with no interior $(d-1-k)$-faces. Thus $f_i(K_n^\circ) = 0$ for all $i \leq d-1-k$. Since $h^c(K_n^\circ)$ is a lower-triangular linear transformation of $f(K_n^\circ)$, this means that $h^c_i(K_n^\circ) = 0$ for all $i \leq d-k$. Thus by Proposition 4.1, $h^c_i(K_n) = 0$ for all $i \geq k$ and so $g^c_i(\partial P_n) = h^c_i(K_n)$ for all $i \leq k$, and $g^c_i(\partial P_n) = 0$ for $k < i \leq \lfloor \frac{d}{2} \rfloor$. Since $f_i(K_n^\circ) = 0$ for $i \leq d-1-k$ and $f_{k-1}(\partial P_n)$ dominates $f_i(\partial P_n)$ for all $i < k-1$, the same is true for $f(K_n)$. Thus $h^c_k(K_n)$ dominates $h^c_i(K_n)$ for all $i < k$. Thus $g^c_k(\partial P_n)$ dominates $g^c_i(\partial P_n)$ for all $i \neq k$. ∎

For cubical 4-polytopes, we can write

$$g^c = (g^c_1, g^c_2) = (f_0 - 16, 16 - 3f_0 + 4f_3).$$

It is easy to verify that the boundary of any 1-stacked 4-polytope has $g^c_2 = 0$, and hence the ray e_1 is in the convex hull of f-vectors of cubical 4-polytopes. The difficulty is in finding cubical 4-polytopes with g^c arbitrarily close to the ray e_2, i.e., cubical 4-polytopes with an arbitrarily high ratio of facets to vertices. Jockusch was able to construct cubical 4-polytopes with a higher ratio of facets to vertices than previously expected possible, but did not determine if there is any bound on this ratio [11].

In general, it is not at all clear whether there exist k-stacked cubical polytopes with many $(k-1)$-faces relative to smaller faces. However for $k = 1$, we have the following, which was noted in [11] for $d = 4$.

COROLLARY 5.6: *For any $n \geq 2^d$ there exists a 1-stacked cubical d-polytope P with at least n vertices, hence $g^c(\partial P)$ lies on the ray e_1, with $g^c_1(\partial P)$ arbitrarily large.*

5.2 PL Cubical Spheres. Though we can go no further with cubical polytopes, we can show that Conjecture 5.2 holds for PL cubical spheres. We prove the following

Theorem 5.7: *For each $1 \leq i \leq d/2$, there exist PL cubical d-spheres with g^c arbitrarily close to the ray e_i.*

Proof: For a simplicial $(d-1)$-polytope P with r vertices, we let C be a PL cubical d-sphere and K a triangulation of P, as given by Theorem 3.1. For $n > 0$, define the iterated fissuring

$$C_P^n := C(MK, C \setminus MK^\circ)^n \tag{12}$$

defined by (4). Since, as in the proof of Theorem 3.1, ∂MK is collared in MK, C_P^n is a PL d-sphere.

By (1) and (5), we compute

$$\begin{aligned} f(C_P^n, t) &= f(C,t) + n(1+t)f(M\partial P, t) \\ &= f(C,t) + n2^r(1+t)f\Big(\partial P, \frac{t}{2}\Big), \end{aligned}$$

and

$$\begin{aligned} g^c(C_P^n, t) &= \frac{t(1-t)^{d+1}}{1+t} f\Big(C_P^n, \frac{2t}{1-t}\Big) + 2^d \frac{(1-t)(1+\tilde{\chi}_{C_P^n}(-t)^{d+2})}{1+t} \\ &= g^c(C,t) + n2^r t(1-t)^d f\Big(\partial P, \frac{t}{1-t}\Big) \\ &= g^c(C,t) + n2^r t g^s(\partial P, t). \end{aligned}$$

Noting that $C = C_P^0$, we rewrite this as

$$g^c(C_P^n, t) = g^c(C_P^0, t) + n2^r t g^s(\partial P, t) \tag{13}$$

to emphasize that the first term is independent of n.

By [13], for fixed d, there are simplicial $(d-1)$-polytopes $P_{m,i}$ such that

$$\frac{1}{g_{i-1}^s(\partial P_{m,i})}\, g^s(\partial P_{m,i}, t) \to t^{i-1} - t^{d+1-i}$$

as $m \to \infty$. Fixing $d \geq 2$ and $1 \leq i \leq d/2$, consider the doubly indexed sequence of cubical d-spheres $C_{P_{m,i}}^n$, indexed by m and n. By a diagonalization argument it follows that for every $k \geq 1$, there are cubical d-spheres $C_{k,i}^d$ such that

$$\frac{1}{g_i^c(C_{k,i}^d)}\, g^c(C_{k,i}^d, t) \to t^i - t^{d+2-i}$$

as $k \to \infty$. This completes the proof. ∎

Remark: For any degree d polynomial $q(t)$, with $q_i = -q_{d-i} \geq 0$ for every $i \leq \lfloor \frac{d}{2} \rfloor$, a similar construction yields cubical spheres S_k and numbers u_k with

$$\frac{1}{u_k}\, g^c(S_k, t) \to tq(t)$$

as $k \to \infty$. This is achieved by noting that the g^c-polynomials of two PL cubical spheres are essentially added by removing one maximal cube from each and identifying the resulting boundaries. (The resulting complex is again a PL sphere [14, Corollary 3.13].) To achieve a fixed ratio $a = q_i/q_j$, for example, we choose the sequences $P_{m,i}$ and $P_{m,j}$ as above and form the corresponding sequences $C^n_{P_{m,i}}$ and $C^n_{P_{m,j}}$. We simultaneously diagonalize these sequences by $n_{i,m} := 2^{r_{mj}} g^s_j(P_{m,j}) q_i k_m$ and $n_{j,m} := 2^{r_{mi}} g^s_i(P_{m,i}) q_j k_m$, attaching $C^{n_{i,m}}_{P_{m,i}}$ and $C^{n_{j,m}}_{P_{m,j}}$ along maximal faces, as above.

6. Further comments

1. Since we have shown that the Adin g-cone is contained in the closure of the convex hull of f-vectors of PL cubical spheres, it would be especially interesting to determine whether the same is true for cubical polytopes (Conjecture 5.2), and conversely, if the Adin g-cone contains the convex hull of f-vectors of cubical polytopes (Conjecture 5.1). In fact, it would be nice to know if any of the results here remain true if *sphere* is replaced by *polytope.* If the fissuring operation as applied in §3 were to preserve shellability, then all of the spheres constructed there could be asserted to be *shellable* rather than just *PL*.

2. It is natural to compare the Adin h-vector and toric h-vector for cubical complexes. Both are invertible linear transformations of the f-vectors of cubical complexes, symmetric for Eulerian cubical complexes, nonnegative for shellable cubical complexes, and satisfy the reciprocity theorem for relative subcomplexes of balls [16]. A major difference is that the Adin h-vector is a lower-triangular linear transformation of the f-vector but the toric h-vector is not.

It would be of interest to determine if the toric g-cone contains the Adin g-cone. (Hetyei [10] has done this for the corresponding h-cones.) Were the toric g-cone not to contain the Adin g-cone, then our work would show that cubical spheres do not satisfy the toric g-theorem, which holds for rational polytopes.

3. Adin's cubical h-vector is a sum of alternating sums of components of the simplicial h-vectors of links of vertices. Is there a convenient interpretation of these alternating sums? Charney and Davis considered such a sum in the context of metric geometry, as did McMullen in his decomposition of the polytope algebra.

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